Improper Integrals (7.6) Continued

Theorem: For $a > 0$

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} a^{1-p}/(p-1) & p > 1 \\ \text{diverges} & p \le 1 \end{cases}$$

Exercise: Check this!

Example: $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{R\to\infty} \int_0^R xe^{-x}$

by parts $(u = x, v' = e^{-x})$ $\displaystyle= \lim_{R\to\infty} -(x+1)e^{-x}\Big|_0^R$

$$= \lim_{R\to\infty} 1 - \frac{R+1}{e^R}$$

$$= 1 - \lim_{R\to\infty} \frac{R+1}{e^R} = 1 - 0$$

(l'Hopital Rule)

Example (Integrands with infinite discontinuities)

1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{R\to 0^+} \int_R^1 \frac{dx}{\sqrt{x}} = \lim_{R\to 0^+} 2\sqrt{x}\Big|_R^1$

$\frac{1}{\sqrt{x}}$ is disc. at $x = 0$

$$= \lim_{R\to 0^+} (2 - 2\sqrt{R}) = 2$$

2) $\int_0^7 \frac{1}{x}\,dx = \lim_{R\to 0^+} \int_R^7 \frac{1}{x}\,dx = \lim_{R\to 0^+} (\ln 7 - \ln R)$

$\frac{1}{x}$ is discont. at 0

$= \ln 7 - \lim_{R\to 0^+} \ln R$

$= \infty$

$\Rightarrow$ The integral diverges.

Comparing Integral

Sometimes we can't (don't know how) evaluate an integral directly but want to know whether or not it converges.

Example: $I = \int_1^\infty e^{-x}/x\,dx$

when $x \geq 1$:

$0 \leq \frac{1}{x} \leq 1$

$\Rightarrow \quad 0 \leq \frac{e^{-x}}{x} \leq e^{-x}$

$\Rightarrow \quad 0 \leq \int_1^\infty e^{-x}/x\,dx \leq \int_1^\infty e^{-x}\,dx = e^{-1}$

(so I converges)

Theorem: (comparison test)

If $0 \le f(x) \le g(x)$ for $x \ge 0$

then: If $g(x)$ converges, $f(x)$ converges

and If $f(x)$ diverges, then $g(x)$ diverges

Example: Does $\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converge?

Sol'n: when $x \ge 1$, $\sqrt{x}+x^2 \ge x^2$

$$\Rightarrow \frac{1}{\sqrt{x}+x^2} \le \frac{1}{x^2}$$

and $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} -\frac{1}{x}\Big|_1^R = 1$

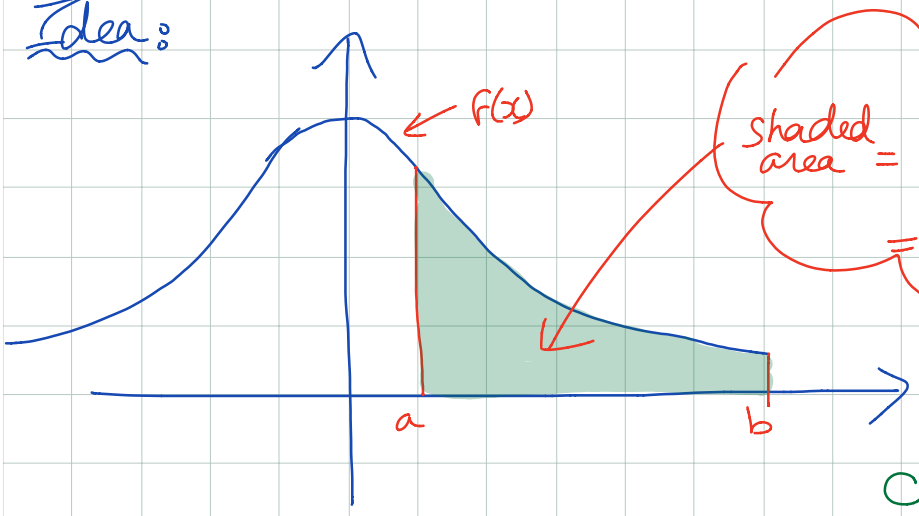
So $\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converges.

7.8) Numerical Integration

Sometimes we can't find an explicit expression for the antiderivative of a function. So to evaluate a definite integral, we have to use numerical methods.

Examples: $\int_1^7 e^{-x^2}\,dx$, $\int_1^9 e^{-x}/x\,dx$

Idea:



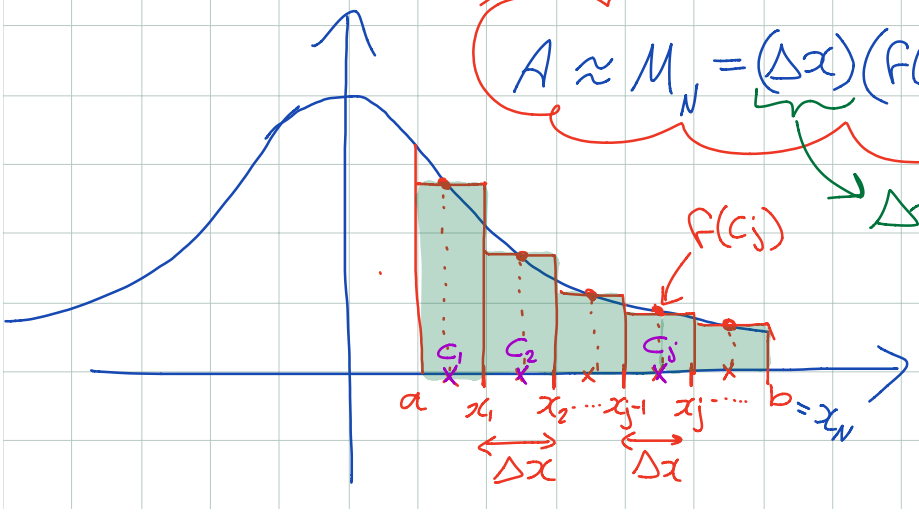
shaded area $= \int_a^b f(x)\,dx = A$

Method 1: Midpoint rule

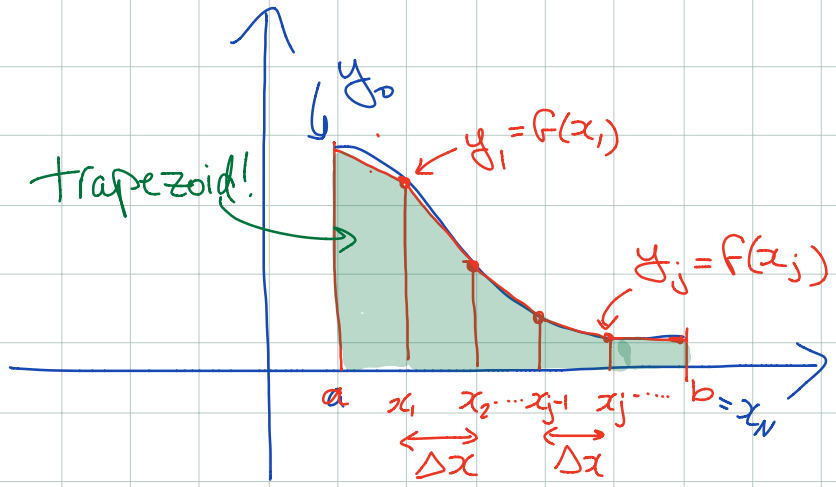
C_j = midpt of j^{th} interval

$$A \approx M_N = (\Delta x)\left(f(C_1) + f(C_2) + \cdots + f(C_N)\right)$$

$$\Delta x = \frac{b-a}{N}$$



Method 2 : Trapezoid Rule



Since the area of a trapezoid is $\frac{(\text{base1}+\text{base2})\times\text{height}}{2}$

we have

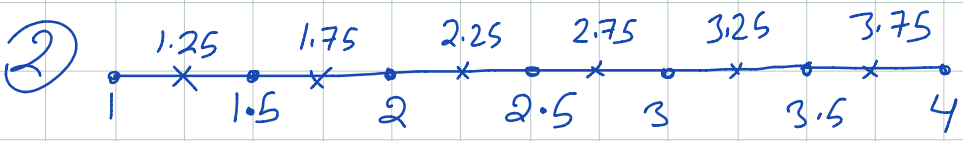
$$A \approx T_N = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots 2y_{N-1} + y_N\right)$$

Example/Exercise :

Calculate T_6 & M_6 for $\int_1^4 \sqrt{x}\,dx$

Solution:

① $\Delta x = \frac{b-a}{N} = \frac{4-1}{6} = \frac{1}{2}$

② 

③ $T_6 = \underbrace{\frac{1}{2}}_{\Delta x}\left(\frac{1}{2}\right)\left(\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}\right)$

≈ 4.661488

$M_6 = \underbrace{\frac{1}{2}}_{\Delta x}\left(\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}\right)$

$= 4.669245$

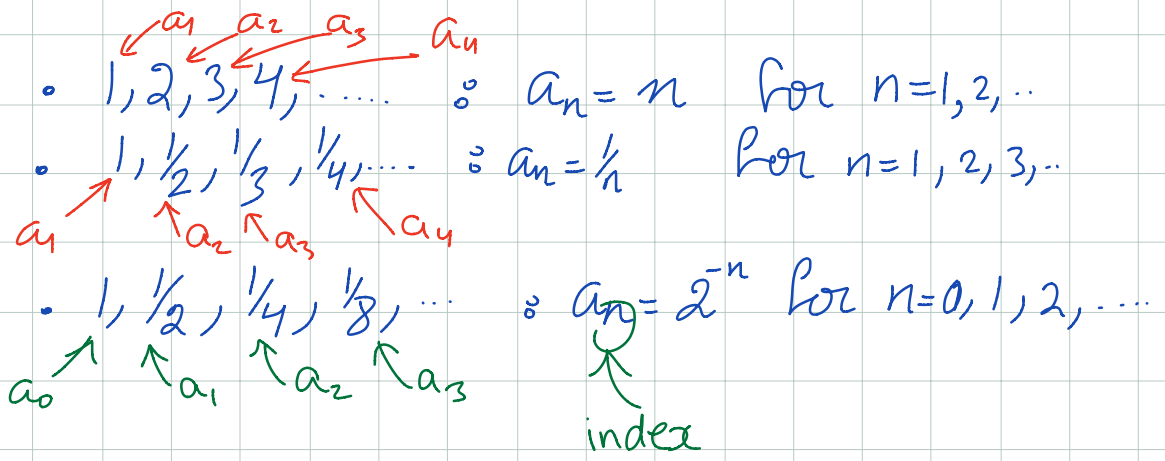
(The exact value is $\int_1^4 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4$

$= 14/3 = 4.6\overline{6}$)

10.1) Sequences:

A sequence is a function $f(n)$ on a set of integers n.

Examples:



Example: (Recursive Sequence)

$$a_1 = 1 \quad , \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

$$\Rightarrow a_1 = 1, \quad a_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.4167$$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) \approx 1.414216$$

This sequence looks like its getting closer & closer to $\sqrt{2}$!

Limit of a Sequence:

We say that a sequence $\{a_n\}$ converges to a limit L, and we write

$$\lim_{n\to\infty} a_n = L \quad (\text{or } a_n \to L)$$

if for every $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ for every $n > N$

(In words, by going far enough in the sequence we can get as close as we like to L).

- If no limit exists, we say $\{a_n\}$ diverges
- If the terms increase without bound, we say $\{a_n\}$ diverges to infinity

Example: Let $a_n = \frac{1}{n}$ $n = 1, 2, \ldots$

What is $\lim_{n\to\infty} a_n$?

Sol'n: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

(exercise: prove this using the definition)

Theorem: If $\lim_{x\to\infty} f(x)$ converges to L then the sequence $a_n = f(n)$ converges to L also.

Example: Find the limit of the sequence

$$a_n = \frac{n^2 - 7}{n^2}, \quad n = 1, 2, \ldots$$

Solution: $\lim_{n\to\infty} \frac{n^2-7}{n^2} = \lim_{x\to\infty} \frac{x^2-7}{x^2} = \lim_{x\to\infty} 1 - \frac{7}{x^2} = 1$

Limit Laws for Sequences

Suppose $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} b_n = M$

Then

- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$
- $\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$
- $\lim_{n\to\infty} (a_n / b_n) = L/M$ (if $M \neq 0$)
- $\lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n = cL$ (for any constant c)

Squeeze theorem:

If $b_n \leq a_n \leq c_n$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} a_n = L$

Example: Find $\lim_{n\to\infty} \frac{\sin(n)}{n}$

Sol'n: $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

and $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

by the Squeeze Theorem.

Example: Find $\lim_{n\to\infty} \frac{R^n}{n!}$ for all $R > 0$

Sol'n: We'll use the Squeeze Theorem

with $0 \leq \frac{R^n}{n!} \leq ?$

to find the upper bound, let's write

$$\frac{R^n}{n!} = \left(\frac{R}{1} \times \frac{R}{2} \times \cdots \times \frac{R}{M}\right)\left(\frac{R}{M+1} \times \frac{R}{M+2} \times \cdots \frac{R}{n-1}\right)\frac{R}{n}$$

M: $M \leq R$, $M+1 \geq R$

$= C$ $\qquad \leq 1$

So $\frac{R^n}{n!} \leq \frac{CR}{n} \xrightarrow[n\to\infty]{} 0$

$$\Rightarrow \lim_{n\to\infty} \frac{R^n}{n!} = 0$$

Limit inside a cont's function

If $f(x)$ is continuous & $\lim_{n\to\infty} a_n = L$

then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Example: Find $\lim_{n\to\infty} \left(\frac{3n}{n+1}\right)^7$

Sol'n: $\lim_{n\to\infty} \left(\frac{3n}{n+1}\right)^7 = \left(\lim_{n\to\infty} \frac{3n}{n+1}\right)^7 = 3^7$

Example: Find $\lim_{n\to\infty} \cos\left(\frac{n}{n^2+1}\right)$

Sol'n: $\lim_{n\to\infty} \cos\left(\frac{n}{n^2+1}\right) = \cos\left(\lim_{n\to\infty} \frac{n}{n^2+1}\right)$

$= \cos(0) = \underline{1}$

Section 10.2 Infinite Series

Q: How does a calculator compute $\sin(x)$ or $\cos(x)$?

Answer: It turns out (we'll see this later) that, for example,

$$\sin(2) = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \cdots$$

Infinite Series

$$\pi = 4 - 4/3 + 4/5 - 4/7 + 4/9 - \cdots$$

Q: But how do we compute these infinite sums?

Answer: We use limits!

Partial Sums: Let $\{a_n\}$ be a sequence.

Its N^{th} partial sum is $\boxed{S_N = a_1 + a_2 + \cdots + a_N}$

Example: (for the sin(1) sequence)

$S_1 = 2$

$S_2 = 2 - 2^3/6 \approx 0.667$

$S_3 = 2 - 2^3/6 + 2^5/120 \approx 0.9333$

$S_4 = 2 - 2^3/6 + 2^5/120 - 2^7/7! \approx 0.9079$

$S_5 = \cdots \approx 0.909347$

$\vdots$

$\lim_{N\to\infty} S_N = \sin(2) \approx 0.909297$

(we'll see that this is true later)

In general, we say an infinite series $\sum_{n=k}^{\infty} a_n$ converges to S if

$$\lim_{N\to\infty} S_N = S.$$

In this case we write $\sum_{n=k}^{\infty} a_n = S$.

(If the limit doesn't exist, we say the series diverges)

Example (Telescoping Series)

Using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution:

We want to find $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)}_{S_N} \quad \text{(by the fact)}$$

Let's write out a few terms

$$S_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$S_3 = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right)$$

$$S_4 = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{5}\right)$$

In general: $S_N = 1 - \frac{1}{N+1} \Rightarrow \lim_{N\to\infty} S_N = 1$

Remark: • A sequence is a "list of numbers" $\{a_n\}$

• A series is their sum

Properties of Infinite Series

If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converge

then $\sum a_n \pm \sum b_n = \sum (a_n \pm b_n)$

$$\sum c a_n = c \sum a_n$$

(c: any constant)

Geometric Series

Example: $\sum_{n=0}^{\infty} (0.5)^n = 0.5^0 + 0.5^1 + 0.5^2 + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$= 2$

In general

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r}$$

c: any constant

$|r|<1$

$$S_N = \sum_{n=0}^{N} cr^n = \frac{c(1-r^{N+1})}{1-r}$$

Example: Evaluate $\sum_{n=0}^{\infty} 3^{-n}$

Soln: $S = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{3}\right)^n}_{r^n} = \frac{1}{1-r} = \frac{1}{1-1/3} = 3/2$

Example: Evaluate $\sum_{n=3}^{\infty} 7(-3/4)^n$

Solution: $S = \sum_{n=3}^{\infty} \underbrace{7}_{c}\underbrace{(-3/4)^n}_{r^n}$

careful! The sum is from $n=3$

So $S = \sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$

How do we fix this?

Let $m = n-3 \Rightarrow n = m+3$

$$S = \sum_{m+3=3}^{\infty} 7\left(-\frac{3}{4}\right)^{m+3} = \sum_{m=0}^{\infty} \underbrace{7\left(-\frac{3}{4}\right)^3}_{c}\underbrace{\left(-\frac{3}{4}\right)^m}_{r^m}$$

$$= 7\left(-\frac{3}{4}\right)^3 \cdot \frac{1}{1-\left(-\frac{3}{4}\right)} = 7\left(-\frac{3}{4}\right)^3 \frac{4}{7} = -\frac{27}{16}$$

In general:

$$\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$$